The n-Queens Problem

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1. INTRODUCTION. The n-queens problem asks how many ways can one put n queens on an $n \times n$ chessboard so that no two queens attack each other. In other words, how many points can be placed on an $n \times n$ grid so that no two are on the same row, column, or diagonal (see Figure 1).

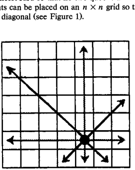

Figure 1. Chess Queen.

This question was first posed for the ordinary 8×8 chessboard as an anonymous problem **[3]**, later attributed to Max Bezzel **[1**, p. 211**]**. The problem received wide attention, however, when posed by Franz Nauck in 1850 **[22]**. Writing about the problem in a letter to the astronomer Schumacher, Gauss conjectured that there were 72 solutions **[9]**. Soon after this 92 solutions were published which convinced Gauss that he had been incorrect. The 92 solutions are commonly represented by 12 "fundamental" solutions, that is, solutions that are not reflections and rotations of each other (see Figure 2). That 92 was the right answer, however, was not proved formally until 1874 by Dr. Glaisher **[10]**, **[27]**, using an idea of Günther (see Section 5). For a history of early results see **[1]** and the bibliography of **[28]**.

These days the 8-queens problem is most often encountered as an exercise in introductory artificial intelligence programming courses. In fact, the n-queens problem is one of the benchmarks by which backtracking algorithms have been compared **[32]**, **[12]**, **[11]**.

For the general $n \times n$ case denote by $Q(n)$ the number of solutions. It is not immediately obvious whether $Q(n) > 0$ for general n and there have been a number of independent proofs showing that $Q(2) = Q(3) = 0$, $Q(n) > 0, n > 3$.

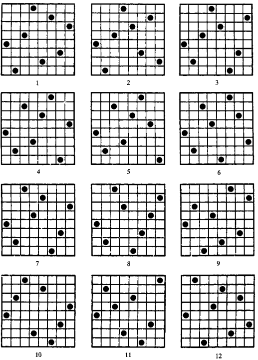

Figure 2. Fundamental solutions to 8 × 8 problem.

The first proof seems to be by Ahrens [1], but proofs by authors unaware of this reference appear in [34], [14], [6]. Other proofs can be found in [28], [5], [23]. It is interesting that none of these notes that $Q(1) = 1$ [35].

The precise nature of $Q(n)$ seems very difficult to understand and a more tractable problem appears to be the *toroidal n-queens problem:* How many ways can one place n-queens on an $n \times n$ chessboard so that no two queens can be on the same row, column, or extended diagonal (see Figure 3). This problem was first studied by Pólya [1, p. 363–374] who showed that $T(n) > 0$ if and only if $(n, 6) = 1$, where $T(n)$ denotes the number of $n \times n$ toroidal queens solutions.

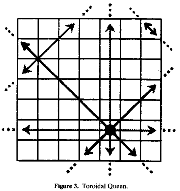

Figure 3. Toroidal Queen.

Finding closed expressions for $Q(n)$ and $T(n)$ seems to be an intractable problem, so our results deal instead with estimating the asymptotic order of these quantities.

First, note that every queen solution is also a rook solution, i.e., no two queens can be on the same column or row, and that each rook solution corresponds to a distinct permutation of $\{1,\dots,n\}$ and there are $n!$ such permutations. It follows that the trivial upper bounds are $T(n) \le Q(n) \le n! < e^{n\log n}$.

It appears that the only previous non-trivial lower bound is the one of Lucas **[17]** stating that $T(p) \ge p(p-3)$ if p is prime. In this paper we will show

Theorem 1.

(a) *Let p be a prime such that* $(p-1)/2$ *is not prime, then* $T(p) > 2^{(p-1)/(2d)}$, *where d be the smallest nontrivial divisor of* $(p-1)/2$. *In particular, if* $p \equiv 1 \pmod 4$, *then* $T(p) > 2^{(p-1)/4}$, *but in general one only has the bound* $T(p) > 2^{\sqrt{(p-1)/2}}$.

(b) *If n is divisible by a prime* $\equiv 1 \pmod 4$ *then* $T(n) > 2^{n/5}$.

Note that the above results also hold for $Q(n)$ since $Q(n) \ge T(n)$, while (b) holds for almost all n, since the set of integers not divisible by a prime $\equiv 1 \pmod 4$ has density zero **[13]**.

We have not been able to find super exponential bounds for $Q(n)$ and $T(n)$ but we believe

Conjecture 1.

$$\lim_{\substack{n\to\infty\\(n,6)=1}} \frac{\log T(n)}{n\log n} = \alpha > 0, \qquad \lim_{n\to\infty} \frac{\log Q(n)}{n\log n} = \beta > 0.$$

Finally, $T(n)$ can also be thought of as the number of arrays of non-negative integers with row, column, and broken diagonals summing to one, i.e., a *pandiagonal magic square* [27] with common sum equal to one. Following ideas of [30] we propose

Conjecture 2. *The generating function* $\sum_{n=1}^{\infty}(T(n)/n!)x^n$ *has a closed form.*

2. SURVEY OF PREVIOUS RESULTS. First, write a queens solution as a function $f(k)$, $k = 0, \dots, n-1$, so that the k'th queen is placed at the $(k, f(k))$ coordinate of the chessboard. It follows immediately that f represents a toroidal solution if and only if $k \mapsto f(k)$, $k \mapsto f(k) + k \pmod n$, and $k \mapsto f(k) - k \pmod n$ are all one to one.

Similarly, $f(k)$ represents a (not necessarily toroidal) queens solution if and only if $k \mapsto f(k)$, $k \mapsto f(k) + k$, and $k \mapsto f(k) - k$ are one to one.

We now present an elegant proof [5], [2], that there is always a queens solution for $n > 3$. The proof splits up according to the residue class of $n \pmod 6$.

(a) If $n = 6m + 1$ or $n = 6m + 5$ then $(n, 6) = 1$ and one lets $f(k)$ be given by $f(k) = 2k \pmod n$. This is clearly a toroidal solution (thus an ordinary solution). Note that this is what one would ordinarily consider as putting queens one "knight's move" apart.

(b) If $n = 6m$ or $n = 6m + 4$ then one takes the solution of (a) for the $(n+1) \times (n+1)$ board and removes the queen in the $(0,0)$ position (i.e., the leftmost column and bottom row). The resulting position is an n solution.

(c) If $n = 6m + 2$ or $n = 6m + 3$ first construct a $6m + 2$ solution as follows: Put a queen at $(k, f(k))$, where

$$f(k) = \begin{cases} 2k + (n-2)/2 \pmod n, & \text{if } 0 \le k \le (n-2)/2 \\ n - 1 - f(n-1-k), & \text{if } n/2 \le k \le n-1 \end{cases}.$$

One checks easily that this is a solution. Note that this is a straightforward generalization of solution (10) in Figure 2 for the 8×8 case.

Since this solution does not have a queen on the main diagonal, one can construct a $6m + 3$ solution by adding a row and column to the edge of the board and putting a queen on the new corner. □

Turning to $T(n)$, we prove Pólya's result characterizing n for which $T(n) > 0$. This is also proved in [5], [19], [2]. An extension is given in [20].

Let $(n, 6) = 1$, then, as before, $f(k) = 2k \pmod n$ is a toroidal solution, so $T(n) > 0$. Conversely, one shows that $T(n) > 0$ implies that n is not divisible by 2 or 3.

Assume that $f(k)$ represents an $n \times n$ toroidal solution so $f(k) - k \pmod n$ is a permutation of $0, \dots, n-1$ and

$$\sum_{k=0}^{n-1} (f(k) - k) \equiv \sum_{k=0}^{n-1} k = \frac{n(n-1)}{2} \pmod n.$$

But this sum is also

$$\sum_{k=0}^{n-1} (f(k) - k) = \sum_{k=0}^{n-1} f(k) - \sum_{k=0}^{n-1} k = 0,$$

since $f(k)$ is a permutation of $0, \ldots, n-1$. Therefore n divides $n(n-1)/2$ and it follows that n is odd.

One similarly shows that n is not divisible by 3 by using the more elaborate sum

$$\sum_{k=0}^{n-1} (f(k) - k)^2 + \sum_{k=0}^{n-1} (f(k) + k)^2 - 4 \sum_{k=0}^{n-1} k^2 \equiv 0 \pmod{n}. \square$$

The next result, also due to Pólya [1], shows how to *compose* solution, i.e., if an $m \times m$ solution and an $n \times n$ solution are given then one tries to construct an $mn \times mn$ solution by placing a copy of the $m \times m$ solution where each queen appears in the $n \times n$ solution (see Figure 4).

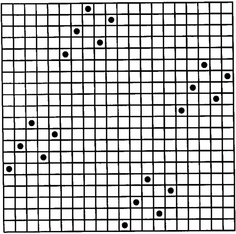

Figure 4. Composed solution.

Let $m, n > 3$, where $(n, 6) = 1$. Then if g is a toroidal $n \times n$ solution and f is an ordinary $m \times m$ solution, then one can compose these to an $mn \times mn$ solution.

Proof: Every integer mod mn can be written uniquely as $an + b, a = 0, \ldots, m - 1, b = 0, \ldots, n - 1$. The claim is that $h(an + b) = f(a)n + g(b)$ is an $mn \times mn$ solution. For example, checking the condition that $h(k) + k$ is one to one:
The assumption

(1) $$h(an + b) + an + b = h(a'n + b') + a'n + b'$$

is seen to imply $g(b) + b \equiv g(b') + b' \pmod{n}$, which gives $b = b'$ since g is a $n \times n$ toroidal solution. Equation (1) then gives that

$$f(a)n + an = f(a')n + a'n,$$

so $a = a'$ since f is a queens solution. The other cases are exactly similar, giving the result.

We now turn to the Lucas estimate $T(p) \geq p(p-3)$ which is proved by noting that for p a prime, each pair $a \neq 0, \pm 1 (\text{mod } p)$ and $b = 0, \dots, p-1$, generates the distinct toroidal solution $f(k) = ak + b$, and the number of such solutions is $p(p-3)$.

We end this section by mentioning results that are tangentially related to the n-queens problem.

(i) Upper bounds for $T(n)$ might be obtained by replacing the toroidal queen with a *toroidal semiqueen*, a piece that moves like a toroidal queen but cannot travel on negative diagonals. The toroidal semiqueen problem can be expressed very simply in terms of permanents and this question was studied by I. Rivin and I. Vardi in **[33, Chapter 6]**.

(ii) A simpler question than the n-queens problem is to compute how many ways one can place k non attacking queens on an $n \times n$ chessboard. Call this number $Q_k(n)$. For $k = 2, 3$ there are closed forms **[1]**, **[28]**,

$$Q_2(n) = \frac{n(n-1)(n-2)(3n-1)}{6},$$

$$Q_3(n) = \frac{(n-1)(n-3)(2n^4 - 12n^3 + 25n^2 - 14n + 1)}{12}.$$

In general, one can show **[31**, Problem 4.15**]** that for fixed k the generating function

$$\sum_{n=1}^{\infty} Q_k(n) x^n$$

is a rational function.

(iii) The ideas in Pólya's paper have been used to give another proof of Fermat's result that every prime $\equiv 1 (\text{mod } 4)$ is a sum of two squares **[16]**.

3. COMPOSING NEW SOLUTIONS. Examining a typical composition constructed by Theorem 3 one notes that there are large areas of the board that have no queens (see Figure 4).

We noted that these regions can be used effectively to construct many more solutions with only a slight variant of the basic composition idea.

Theorem 2. *Let $m, n > 3$ be given, $(n, 6) = 1$, and let $f_1, f_2, \dots, f_{Q(m)}$ be all $m \times m$ queens solutions, and let g be a toroidal $n \times n$ solution. Then for each map π: $\{0, \dots, n-1\} \to \{1, \dots, Q(m)\}$ the function $h(an + b) = f_{\pi(b)}(a)n + g(b)$ gives a distinct $mn \times mn$ solution.*

Proof: The proof that each of these gives a queens solution is exactly as in the previous section, while the fact that each solution is distinct is clear from the definition. □

Corollary 1. *Let $(n, 6) = 1$, $m \geq 3$ then $Q(mn) > [Q(m)]^n T(n)$. In particular, if N is a number divisible by 5, and $(N, 6) = 1$, then $Q(N) > 4^{N/5}$.*

Proof: The first part follows directly from counting the number of solutions generated by Theorem 2. The second result follows by letting $n = 5, m = N/5$, and noting that $Q(5) = 4$. One then checks the special cases $N = 10, 15$. □

Remark. Corollary 1 gives the first example of a set of n's for which $Q(n)$ grows faster than a polynomial in n.

4. USING THE MULTIPLICATIVE STRUCTURE OF Z/pZ. In the previous section we constructed $mn \times mn$ solutions out of $m \times m$ solutions and $n \times n$ solutions. Such a method will not work for the case of $p \times p$ chessboards, when p is prime, so constructing an exponential number of solutions in this case requires a new idea.

The basic idea of composition was to generate solutions using an additive subgroup of Z/nZ. We will now use the multiplicative structure by constructing "quasi-linear" solutions of the form $f(k) = c_k k$, where c_k is constant on cosets of a multiplicative subgroup of $(Z/pZ)^*$.

The simplest case is when $p \equiv 1 \pmod 4$. It is well known **[13**, page 85] that for such p there is a number i mod p with the property that $i^2 \equiv -1 \pmod p$.

Consider an equivalence relation on $\{1, \dots, p-1\}$ by $a \sim b$ if $a = i^k b$, for some k. This defines $(p-1)/4$ equivalence classes $\langle a_1 \rangle, \langle a_2 \rangle, \dots, \langle a_{(p-1)/4} \rangle$ for some sequence numbers $0 < a_1, \dots, a_{(p-1)/4} < p$.

Now consider $f(k) = c_k k$, where $c_k = i$ or $c_k = 1/i$, and c_k is constant on each set $\langle k \rangle$. The claim is that $f(k)$ is a toroidal queens solution.

Theorem 3 (i). *Each map $\sigma: \{1, 2, \dots, (p-1)/4\} \to \{\pm 1\}$ yields a distinct toroidal $p \times p$ solution.*

Proof: For each σ identify $\{1, \dots, (p-1)/4\}$ with the distinct classes $\langle a_1 \rangle, \dots, \langle a_{(p-1)/4} \rangle$ and define f by $f(k) = i^{\sigma(\langle k \rangle)} k$.

To see that f is a toroidal solution we check the three conditions of Section 2:

(a) Assume that $f(k) = f(k')$, then

$$i^{\sigma(\langle k \rangle)} k = i^{\sigma(\langle k' \rangle)} k' \Rightarrow k = i^{\pm 1} k' \Rightarrow \langle k \rangle = \langle k' \rangle \Rightarrow k = k'$$

so f is one to one.

(b) Assume that $f(k) + k = f(k') + k'$ then $i^{\sigma(\langle k \rangle)} k + k = i^{\sigma(\langle k' \rangle)} k' + k'$ so

$$k(1 + i^{\pm 1}) = k'(1 + i^{\mp 1})$$

for one of four choices of $\pm 1, \mp 1$. All these cases lead to $\langle k \rangle = \langle k' \rangle$, for example, if $k(1 + 1/i) = k'(1 + i)$ then the identity $1 + 1/i = (1 + i)/i$ gives $k = ik'$. It follows that $\langle k \rangle = \langle k' \rangle$ so $k = k'$.

(c) If $f(k) - k = f(k') - k'$, then one gets that $k = k'$ as in part (b). The only difference is that $i^2 \equiv -1 \pmod p$ is needed to take case of the case when $k(i - 1) = k'(1/i - 1)$ and $k(1/i - 1) = k'(i - 1)$.

Finally, it is routine to check that for each distinct σ one gets a distinct f. □

In the general case let q be the smallest divisor of $p - 1$ that is even and greater than two. It is known **[13]** that $x^q = 1$ has q solutions mod p and that these form a cyclic group. Let ξ be a generator and, as before, define an equivalence relation by $a \sim b$ if $a = \xi^k b$ for some k. This gives $(p-1)/q$ equivalence classes $\langle a_1 \rangle, \dots, \langle a_{(p-1)/q} \rangle$, where $\langle a \rangle$ represents the equivalence class of a (i.e., $\{a, \xi a, \xi^2 a, \dots, \xi^{q-1} a\}$). The result is

Theorem 3 (ii). *Each map $\sigma: \{1, \dots, (p-1)/q\} \to \{\pm 1\}$ leads to a distinct toroidal $p \times p$ solution.*

Proof: The proof proceeds exactly as in part (i) of the theorem. It is important to note that as in part (c) above, one needs to have $\xi^j = -1$ for some j. This explains why q must be chosen to be an *even* divisor of $p - 1$. □

Theorem 1(a) now follows immediately by counting the number of solutions generated by Theorem 3.

Remark 1. We have found no improvement on the lower bound $p(p-3)$ for primes of the form $p = 2q + 1, q$ prime (Cunningham primes).

Remark 2. For a given q dividing $p - 1$, the solutions constructed by this method have the same cycle structure when taken as permutations, i.e., a product of $(p-1)/q$ cycles of length q.

Remark 3. This method of constructing solutions can be used to give more complicated forms of compositions of solutions. For simplicity consider two distinct primes $p_1, p_2 \equiv 1 (\mathrm{mod}\ 4)$ (the general case is similar).

Theorem 4. *For each map* σ: $\{1, \dots, (p_1 - 1)/4\} \to \{1, \dots, T(p_2)\}$ *and* π: $\{0, \dots, p_2 - 1\} \to \{1, \dots, 2^{(p-1)/4}\}$ *there is a distinct* $p_1 p_2 \times p_1 p_2$ *toroidal solution.*

Proof: Let $f_1, \dots, f_{T(p_2)}$ be the toroidal $p_2 \times p_2$ solutions and $g_1, \dots, g_{2(p_1-1)/4}$ be the solutions as constructed in Theorem 6 (i). One can write each number (mod $p_1 p_2$) uniquely as $ap_1 + bp_2$ where $a = 0, \dots, p_2 - 1, b = 0, \dots, p_1 - 1$. It can then be shown that for each σ, π the function

$$h(ap_1 + bp_2) = f_\sigma^{(\langle b \rangle)}(a) p_1 + g_{\pi(a)}(b) p_2.$$

gives a distinct toroidal solution. □

Counting the number of solutions generated by Theorem 4, one gets that

$$T(p_1 p_2) \geq T(p_2)^{(p_1-1)/4} 2^{p_2(p_1-1)/4}.$$

This can be extended to more complicated compositions for products of more than two primes. A computation shows that in the limit this gives the lower bound $T(n) > 2^{(1-\varepsilon)n/3}$, where $\varepsilon \to 0$ as the number of prime factors of n that are $\equiv 1$ (mod 4) goes to infinity.

We now turn to the proof of Theorem 1 (b). Consider n, $(n, 6) = 1$, and n is divisible by a prime $p \equiv 1$ (mod 4). It follows from Corollary 1 and $p \geq 5$ that

$$T(n) = T((n/p)p) > T(p)^{n/p} T(n/p) > 2^{n(p-1)/(4p)} \geq 2^{n/5}. \quad \square$$

Remark. The reader may have noted that our techniques have been elementary. It is an interesting question why nontrivial lower bounds have escaped the large literature on this subject. We believe that there are two reasons for this.

The first comes for the original formulation of the problem on the 8×8 chessboard. Note that in the proof of Theorem 1 the hardest case was for numbers $\equiv 2$ (mod 6), since they cannot be reached from the more tractable toroidal problem. This might also explain why the toroidal problem has not been extensively investigated.

The second reason is the emphasis on classifying "fundamental" solutions, i.e., solutions that are not rotations or reflections of each other. This problem is difficult even in the much simpler case of the rook's problem, and counting the number of fundamental solutions has proved to be nontrivial **[18]**, **[4]**, **[21]**, **[26]**. Note that fundamental solutions under toroidal symmetries for the toroidal case are very easy to classify—they are the ones with a queen at $(0, 0)$.

5. COMPUTATIONAL RESULTS. As mentioned in the introduction there has been much interest in the computation of $Q(n)$ and $T(n)$ using backtracking algorithms. A different algorithm has been advanced by Igor Rivin and Ramin Zabih **[24]**. Their idea is similar to a method proposed by Günther: Consider independent variables $X_0, \dots, X_{2n-2}, Y_{-n+1}, \dots, Y_{n-1}$ and the matrix $\|X_{i+j}Y_{i-j}\|$, then the squarefree term in X, Y of the *permanent* of this matrix (determinant with no minus signs) gives the number of queens solutions. As before the toroidal case is much cleaner. One has $2n$ variables and considers the squarefree term of the permanent of $\|X_{i+j \bmod n} Y_{i-j \bmod n}\|$.

To estimate the running time of this method in the toroidal case, note that the standard methods for evaluating a permanent give a running time of about 2^n multiplications, and the terms are squarefree polynomials in $2n$ variables and of degree $\leq 2n$. It follows that there are at most 2^{2n} terms. Since a multiplication takes time about 8^n, this gives a running time on the order of 16^n, but with space requirement of 4^n (the running time can be reduced to 8^n **[25]**).

Since backtracking algorithms generate all solutions these take at least $T(n)$ steps to compute $T(n)$. It follows that a lower bound $T(n) > \gamma^n$, where $\gamma > 8$, would show that the Rivin-Zabih algorithm always runs faster than backtracking (if Conjecture 1 holds, then this algorithm will be much faster than backtracking).

On the other hand, backtracking takes very little space so it is still the more practical method for computing large values of $T(n)$ and $Q(n)$, and all the values in Figure 5 were computed this way ($Q(19)$ and $Q(20)$ were computed by A. Shapira **[29]**). For example, the value $T(23) = 128850048$ was computed using backtracking and a number of implementation shortcuts. Toroidal symmetries

n	$T(n)$	$\log T(n)/(n \log n)$	$Q(n)$	$\log Q(n)/(n \log n)$
4			2	0.125
5	10	0.286	10	0.286
6			4	0.129
7	28	0.245	40	0.271
8			92	0.272
9			352	0.297
10			724	0.286
11	88	0.170	2680	0.299
12			14200	0.321
13	4524	0.252	73712	0.336
14			365596	0.347
15			2279184	0.360
16			14772512	0.372
17	140692	0.246	95815104	0.382
18			666090624	0.391
19	820496	0.243	4968057848	0.399
20			39029188884	0.407
23	128850048	0.259		

Figure 5. Values of $T(n)$, $Q(n)$.

reduced the number to be computed to 1482252 solutions. A further saving was to eliminate impossible consecutive triplets. The computation was done in LeLisp as a distributed computation over a network of 20 Suns at INRIA, Rocquencourt, and took 267 days of CPU time.

Note that the computational evidence supports Conjecture 1 since the values of $\log Q(n)/(n \log n)$ and $\log T(n)/(n \log n)$ seem to be monotonically increasing.

REFERENCES

1. W. Ahrens, *Mathematische Unterhaltungen und Spiele, Vol. 1*, B. G. Teubner, Leipzig 1921.
2. J. D. Beasly, *The Mathematics of Games*, Oxford University Press, Oxford 1990.
3. Berliner Schachgesellschaft, **3** (1848), p. 363.
4. S. Chowla, I. N. Herstein, and K. Moore, *On recursions connected with symmetric groups I*, Canadian J. of Math. **3** (1951), 328–334.
5. D. Clark, *A combinatorial theorem on circulant matrices*, This Monthly **92** (1985), 725–729.
6. B.-J. Falkowski and L. Schmitz, *A note on the queen's problem*, I.P.L. (= Information Processing Letters) **23** (1986), 39–46.
7. L. R. Foulds and D. G. Johnston, *An application of Graph Theory and Integer Programming: Chessboard non-attacking puzzles*, Math. Gazette **57** (1984), 95–104.
8. M. Gardner, *The unexpected Hanging and other Mathematical Diversions*, Simon and Schuster, New York 1986.
9. C. F. Gauss, Letters to H. C. Schumacher, September 12–27, 1850, *Werke, Vol. 12*, Springer, Berlin 1929, p. 20–28.
10. Dr. J. W. L. Glaisher, Philosophical Magazine, **18** (1874), 457–467.
11. S. Golomb and L. Baumert, *Backtrack programming*, J.A.C.M. **12** (1965), 516–524.
12. R. Haralick and G. Elliott, *Increasing Tree Search Efficiency for Constraint Satisfaction Problems*, Artificial Intelligence **14** (1980), 263–313.
13. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford 1988.
14. E. J. Hoffman, J. C. Loessi, and R. C. Moore, *Construction for the solution of the n-queens problem*, Math. Mag. **42** (1969), 66–72.
15. M. Kraitchik, *Mathematical Recreations*, Dover, New York 1953.
16. L. Larson, *A theorem about primes proved on a Chessboard*, Math. Mag. **50** (1977), 69–74.
17. E. Lucas, *Récréations Mathématiques, Vol. I*, Albert Blanchard, Paris 1977.
18. E. Lucas, *Théorie des Nombres*, Albert Blanchard, Paris 1961.
19. P. Monsky, Problem E 2698, This Monthly **85** (1978), p. 116.
20. P. Monsky, Problem E 3162, This Monthly **93** (1986), p. 566.
21. L. Moser and M. Wyman, *On the solution of* $x^d = 1$ *in symmetric groups*, Canadian J. of Math. **7** (1955), 159–168.
22. F. Nauck, *Illustrierten Zeitung*, **14**, p. 352, June 1, 1850; **15**, p. 182, September 21, 1850; **15**, p. 207, Semptember 28, 1850.
23. M. Reichling, *A Simplified Solution of the N Queens' Problem*, I.P.L. **25** (1987), 253–255.
24. I. Rivin and R. Zabih, *An Algebraic Approach to Constraint Satisfaction Problems*, Proceedings of the Int. Joint Conf. on A.I., Detroit 1989.
25. I. Rivin and R. Zabih, *A dynamic programming solution to the N-queens problem*, I.P.L. **41** (1992), 253–256.
26. R. W. Robinson, *Counting the arrangements of Bishops*, in "Combinatorial Mathematics IV, Adelaide 1975," L. Notes in Math. **560** (1976), Springer-Verlag 1976, 198–214.
27. W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, Dover 1987.
28. A. Sainte-Laguë, *Les Réseaux (ou Graphes)*, Mém. Des Sc. Math. **18**, Ghautier-Villars, Paris 1926.
29. A. Shapira, Personal communication, August 1992.
30. R. P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser 1983.
31. R. P. Stanley, *Enumerative Combinatorics, Vol. 1*, Wadsworth 1986.
32. H. S. Stone and J. M. Stone, *Efficient Search Techniques-An Empirical study of the N-Queens Problem*, IBM Technical Report, T. J. Watson Research Center.
33. I. Vardi, *Computational Recreations in Mathematica*, Addison Wesley 1990.

34. A. M. Yaglom and I. M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions, Vol. I*, Dover 1987.
35. R. Zabih, Personal communication 1988.

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The Prince of Algebra

Madam Professor,
Let me introduce myself—
I'm Albert James,
whom you may know
by my test score
that's lower than my age.

Your algebra tests
are too long for me
in fifty minutes,
but I am proud
of my attendance—
I never miss class,
never come late.

I am preparing
for a new career.
For thirty years I was
with the Postal Service
never absent,
never late.

Your mathematics
is important!
It runs the clock
which runs the mail.
Now I train to be
a first grade teacher.

I will teach
mathematics
by punctuality
and perfect attendance.

From *Intersections: Poems by JoAnne Growney, Kadet Press, Bloomsburg, PA*, 1993, *p.* 52–53.